Does multifractal theory of turbulence have logarithms in the scaling relations?

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The multifractal theory of turbulence uses a saddle-point evaluation in determining the power-law behaviour of structure functions. Without suitable precautions, this could lead to the presence of logarithmic corrections, thereby violating known exact relations such as the four-fifths law. Using the theory of large deviations applied to the random multiplicative model of turbulence and calculating subdominant terms, we explain here why such corrections cannot be present.

1. Introduction

In fully developed turbulence there is now fairly good evidence for anomalous scaling, that is scaling exponents which cannot be predicted by dimensional analysis. Some of this evidence is reviewed in Frisch (1995). This reference also contains a detailed presentation of the multifractal formalism in the formulation of Parisi & Frisch (1985) (see also Benzi *et al.* 1984). In this formalism, anomalous scaling for structure functions (moments of velocity increments) is connected by a Legendre transformation to the distribution of singularities of the velocity field. An earlier and alternative formalism for anomalous scaling was introduced by the Russian School of Kolmogorov (Obukhov 1962; Kolmogorov 1962; Yaglom 1966). In its simplest version it uses a random multiplicative model for calculating the statistical fluctuations of the energy dissipation on various scales; the fractal properties of this model were discovered by Mandelbrot (1974). The bridging of the two formalisms is discussed in Frisch (1995) in the light of the theory of large deviations for the sums of independent identically distributed random variables, discovered in the 1930s by Cramér (1938).

We recall that Parisi & Frisch's original formulation gives an integral representation of the structure functions which are then evaluated by the method of steepest descent through a saddle point. Specifically, the structure function of order p for a separation ℓ is given by

$$\frac{S_p(\ell)}{v_0^p} \sim \int \mathrm{d}\mu(h) \left(\frac{\ell}{\ell_0}\right)^{ph+3-D(h)}. \tag{1.1}$$

We have here used the notation of Frisch (1995): v_0 is the r.m.s. velocity fluctuation, ℓ_0 is the integral scale, $D(h)$ is the fractal dimension associated with singularities of scaling exponent h and $\mathrm{d}\mu(h)$ gives the weight of the different exponents. For a given p, let us assume that the exponent $ph+3-D(h)$ has a minimum ζ_p, as a function

of h, and that it behaves quadratically near this minimum. Standard application of Laplace's method of steepest descent (see e.g. Bender & Orszag 1978) shows then that, at small separations, $S_p(\ell)$ varies as $(\ell/\ell_0)^{\zeta_p}$ but with a logarithmic prefactor $[\ln(\ell/\ell_0)]^{-1/2}$ stemming from the Gaussian integration near the minimum. For $p = 3$ this logarithmic prefactor is clearly inconsistent with the four-fifths law of Kolmogorov (1941), one of the very few exact results in high-Reynolds-number turbulence, which tells us that the third-order (longitudinal) structure function is given by $-(4/5)\varepsilon\ell$, where ε is the mean energy dissipation per unit mass.

In Frisch (1995) this difficulty was handled by writing

$$\lim_{\ell \to 0} \frac{\ln S_p(\ell)}{\ln \ell} = \zeta_p. \tag{1.2}$$

Indeed, by taking the logarithm of the structure function we change the *multiplicative* logarithmic correction into an *additive* log-log correction which, after division by $\ln \ell$, becomes *subdominant* as $\ell \to 0$. But if we do not take the logarithm of the structure function, is there a logarithmic correction in the leading term whose presence, for $p = 3$, would invalidate the standard multifractal formalism?

It is also well known that such logarithmic corrections are absent in the random multiplicative model, a matter we shall come back to. Actually, as we shall see, the random multiplicative model gives the key that allows us to understand why logarithmic corrections are unlikely and are definitely ruled out in the third-order structure function.

In §2 we recall some basic facts about the random multiplicative model and the way it can be reformulated in terms of multifractal singularities using large-deviations theory. This is the theory introduced by Cramér (1938) which allows the estimation of the (very small) probability that the sum of a large number of random variables deviates strongly from the law of large numbers.† In this way the random multiplicative model can be reformulated in standard multifractal language; naive application of PF would then suggest the presence of logarithmic corrections. In §3 we show how a refined version of large-deviations theory, which goes beyond leading order, removes the logarithmic corrections, which are cancelled by other logarithmic corrections in the probability densities. In §4 we summarize our findings, return to the general multifractal formalism beyond the specific random multiplicative model and show that the four-fifths law allows us to obtain the first subleading correction to the usual multifractal probability.

2. The random multiplicative model

In the random multiplicative model (see e.g. Frisch 1995, for details) one assumes that an integral-scale size cube with side ℓ_0 is subdivided into 8 first-level cubes of half the side, which in turn are divided into 8^2 second-level cubes of side $\ell_0 2^{-2}$, and so on. The 'local' dissipation at the nth level with scale $\ell = \ell_0 2^{-n}$ is defined as

$$\varepsilon_\ell = \varepsilon W_1 W_2 \cdots W_n, \tag{2.1}$$

where ε is a non-random mean dissipation per unit mass and the W_i are positive, independently and identically distributed random variables of unit mean value.

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† The reader is only assumed to be familiar with a very elementary version of large-deviations theory, as explained e.g. in §8.6.4. of Frisch (1995); a more detailed exposition for physicists interested e.g. in the foundations of thermodynamics can be found in Lanford (1973); Varadhan (1984) and Dembo & Zeitouni (1998) are written for the more mathematically minded reader.

The ensemble average $\langle \varepsilon_\ell \rangle$ is thus still equal to ε, so that the cascade is conservative only in the mean. Since we are interested in describing the inertial-range scaling properties ($\ell \ll \ell_0$), we shall mainly focus our attention on high-order generations, i.e. on large values of n, where large fluctuations of ε_ℓ are present. As a consequence, the formalism of multiplicative variables leads to the presence of very large fluctuations. The correspondence between multifractality and the probabilistic theory is expressed by the relationship

$$n = -\log_2 \frac{\ell}{\ell_0} = -\frac{1}{\ln 2} \ln \frac{\ell}{\ell_0}. \tag{2.2}$$

Thus n can be viewed either as the number of W factors determining the local dissipation or as the number of cascade steps leading from the 'injection' length scale ℓ_0 to the current scale ℓ.

It is elementary and standard to show that the moments of the local dissipation are given by

$$T_p(\ell) \equiv \langle \varepsilon_l^p \rangle = \langle (\varepsilon W_1 \cdots W_n)^p \rangle = \varepsilon^p \langle W^p \rangle^n = \varepsilon^p \left(\frac{\ell}{\ell_0} \right)^{-\log_2 \langle W^p \rangle}. \tag{2.3}$$

Then, following the suggestion originally made by Obukhov (1962), one calculates the structure functions at separation ℓ by the Kolmogorov (1941) expression in which one replaces the mean dissipation by its local random value ε_ℓ, to obtain

$$S_p(\ell) \approx \langle (\ell \varepsilon_\ell)^{p/3} \rangle = l^{p/3} T_{p/3}(\ell) \propto \ell^{\zeta_p}, \tag{2.4}$$

where $\zeta_p = p/3 - \log_2 \langle W^{p/3} \rangle$. Obviously the third-order structure function has exponent unity, as required by the four-fifths law and none of the structure functions has any multiplicative logarithmic factor.

There is however an alternative and somewhat roundabout way of evaluating the structure functions for the random multiplicative model. First we transform the product of positive random variables into a sum by setting $W_i = 2^{-m_i}$, to obtain

$$T_p(\ell) = \langle (\varepsilon 2^{-m_1} \cdots 2^{-m_n})^p \rangle = \varepsilon^p \langle 2^{-m_1 p} \cdots 2^{-m_n p} \rangle$$
$$= \varepsilon^p \langle 2^{-nxp} \rangle = \varepsilon^p \int \mathrm{d}x \, \mathrm{e}^{-nxp \ln 2} P_n(x), \tag{2.5}$$

where

$$x \equiv \frac{m_1 + \ldots + m_n}{n} \tag{2.6}$$

is the sample mean of n independent and identically distributed random variables and $P_n(x)$ its probability density function (PDF). When the PDF $P(m)$ of the individual variables m_i falls off very quickly at large arguments, as is usually assumed in the random multiplicative model (here we assume that $P(m)$ falls off faster than exponentially at large $|m|$), all the moments of the m_i are finite and the law of large numbers (Feller 1968) implies that $P_n(x)$ is, for large n, increasingly concentrated near the mean $\langle m \rangle$. The theory of large deviations (Cramér 1938) tells us roughly that when $x \neq \langle m \rangle$ its probability falls off exponentially with n as $\mathrm{e}^{ns(x)}$, where the Cramér function $s(x)$ is non-positive up-convex and vanishes at $x = \langle m \rangle$. The Cramér function can be expressed as the Legendre transform

$$s(x) = \inf_\alpha [\alpha x + \ln Z(\alpha)] \tag{2.7}$$

of the characteristic function

$$Z(\alpha) \equiv \int \mathrm{d}y \, \mathrm{e}^{-\alpha y} P(y) = \langle \mathrm{e}^{-\alpha m} \rangle. \tag{2.8}$$

The correct statement of large deviations is that $\ln P_n(x)/n$ tends to $s(x)$ as $n \to \infty$. Suppose however we somewhat sloppily write the large-deviations result as

$$P_n(x) \sim \mathrm{e}^{ns(x)} \quad \text{(incorrect)} \tag{2.9}$$

and use this in (2.5). We then obtain an integral representation for $T_p(\ell)$ and thus for $S_p(\ell)$ which when evaluated by steepest descent for large n will give not just a power law in ℓ but also a multiplicative correction proportional to $1/\sqrt{-\ln(\ell/\ell_0)}$, thus contradicting (2.4).

   To resolve the paradox we need to extend the large-deviations result beyond the leading order. This is called the theory of refined large deviations, first developed by Bahadur & Ranga Rao (1960) and which is reviewed in Dembo & Zeitouni (1998). In the next section we shall show how this can be done by rather elementary application of steepest descent.

3. Refined large-deviations theory and the disappearance of logs

   We now derive the asymptotic expansion for large n of the PDF $P_n(x)$ of the sample mean (2.6). Consider the characteristic function of the sample mean

$$Z_n(\alpha) \equiv \int \mathrm{d}x \, \mathrm{e}^{-\alpha x} P_n(x) = \left\langle \mathrm{e}^{-\alpha(m_1 + \ldots + m_n)/n} \right\rangle \tag{3.1}$$

$$= \left\langle \mathrm{e}^{-\alpha m_1/n} \cdots \mathrm{e}^{-\alpha m_n/n} \right\rangle = \left\langle \mathrm{e}^{-\alpha m/n} \right\rangle^n = Z^n\left(\frac{\alpha}{n}\right), \tag{3.2}$$

where $Z(\alpha)$ is the characteristic function for a single variable m, defined in (2.8). Since we assumed that $P(m)$ falls off faster than exponentially, $Z(\alpha)$ can be defined for any complex α. Hence we can invert the Laplace transform appearing in (3.1) by a Fourier integral along a contour C running from $-\mathrm{i}\infty$ to $+\mathrm{i}\infty$ (see figure 1)

$$P_n(x) = \frac{1}{2\mathrm{i}\pi} \int_C \mathrm{d}\beta \, \mathrm{e}^{\beta x} Z^n\left(\frac{\beta}{n}\right). \tag{3.3}$$

We recast (3.3) in exponential form

$$P_n(x) = \frac{1}{2\mathrm{i}\pi} \int_C \mathrm{d}\beta \, \mathrm{e}^{\beta x + n \ln Z(\beta/n)} = \frac{n}{2\mathrm{i}\pi} \int_C \mathrm{d}\gamma \, \mathrm{e}^{n[\gamma x + \ln Z(\gamma)]}, \tag{3.4}$$

with the substitution $\gamma = \beta/n$. By (2.7) the argument of the exponential has a minimum $s(x)$ along the real γ-line at a point $\alpha_\star(x)$. Taking the contour C through $\alpha_\star(x)$, the argument of the exponential will now have a maximum at this point; thus (3.4) can be evaluated by steepest descent (see e.g. Bender & Orszag 1978).

   We recall that for an integral of the form

$$I(n) = \int_C \mathrm{d}y \, f(y) \mathrm{e}^{n\phi(y)}, \tag{3.5}$$

with a saddle point $y_\star$ where ϕ' vanishes and where neither f nor ϕ'' vanish, the large-n behaviour is given by

$$I(n) = \sqrt{\frac{2\pi}{-n\phi''(y_\star)}} \, \mathrm{e}^{n\phi(y_\star)} f(y_\star) \left[1 + O\left(\frac{1}{n}\right)\right]. \tag{3.6}$$
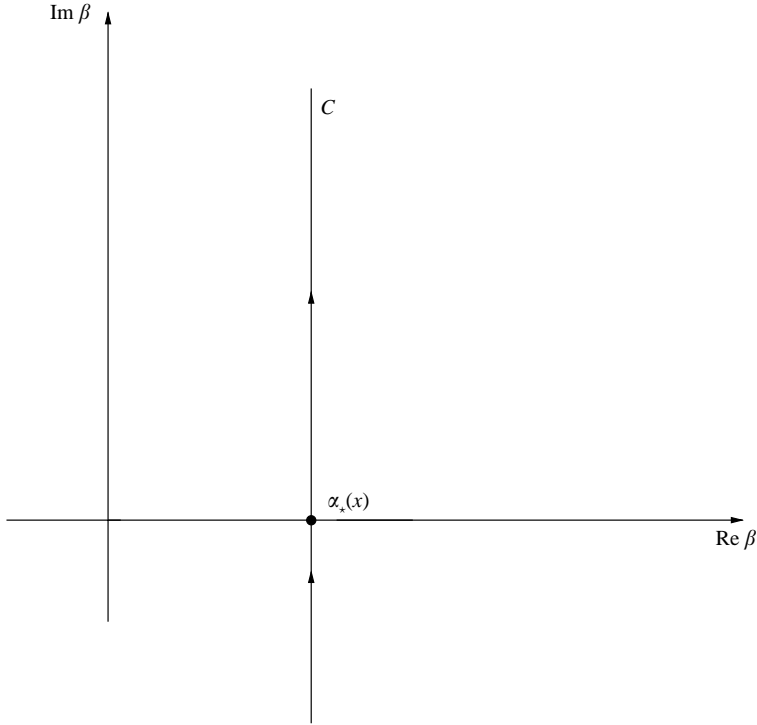
FIGURE 1. The integration contour C through the saddle $\alpha_\star$ in the complex β-plane.

The saddle point formula (3.6), applied to (3.4) gives, after taking a logarithm

$$\frac{\ln P_n(x)}{n} = s(x) + \frac{\ln n}{2n} - \frac{\ln(2\pi\,Q)}{2n} + O\left(\frac{1}{n^2}\right), \tag{3.7}$$

where $Q > 0$ is the second derivative of $\alpha x + \ln Z(\alpha)$, evaluated at the saddle point $\alpha_\star(x)$. As n does not appear in Q, which is solely a function of x, the right-hand side of (3.7) is thus structured as an inverse power series in n, except for the first subleading term which contains a logarithm.

Note that expressions such as (3.7) are very common in thermodynamic applications of large deviations when dealing with the logarithm of the (very large) number of states (see e.g. Lanford 1973).

We can of course rewrite (3.7) in exponential form as

$$P_n(x) = \sqrt{n/(2\pi\,Q)}\,\mathrm{e}^{ns(x)}\left[1 + O\left(\frac{1}{n}\right)\right]. \tag{3.8}$$

In this way we see that $P_n(x)$ has a multiplicative $\sqrt{n}$ correction. Recalling that in the random multiplicative model $n = -\log_2(\ell/\ell_0)$, this correction is just what we need to cancel the unwanted logarithms in the structure functions obtained when the incorrect form (2.9) is used.

It is important to note that the quantity which goes to a finite limit for large n is $\ln P_n(x)/n$ and that for this quantity the correction we have determined is an additive subleading term. This is why such terms should be regarded as subleading. Of course the cancellation of logarithms for the random multiplicative model cannot take place just at the first subleading order, since (2.4) is an exact expression and has no

logarithms. To evaluate refined large deviations to all orders for a general random multiplicative model is quite cumbersome and we shall not attempt it here because it would not shed additional light on the issue discussed. It can however be done quite easily for simple random multiplicative models such as the black-and-white model of Novikov & Stewart (1964).

4. Back to multifractal turbulence

In multifractal language, the result obtained within the framework of the random multiplicative model is that the probability $P(\ell, h)$ to be within a distance ℓ of the set carrying singularities of scaling exponent between h and $h + dh$ is not $(\ell/\ell_0)^{3-D(h)}d\mu(h)$ but is given, for small ℓ, by

$$P(\ell, h) \propto \left(\frac{\ell}{\ell_0}\right)^{3-D(h)} \left[-\ln \frac{\ell}{\ell_0}\right]^{1/2} d\mu(h), \qquad (4.1)$$

which has a subleading logarithmic correction. We recall that it must be qualified 'subleading' because the correct statement of the large-deviations leading-order result involves the logarithm of the probability divided by the logarithm of the scale. The correction is then a subleading additive term.

It is important to mention that the presence of a square root of a logarithm correction in the multifractal probability density has already been proposed by Meneveau & Sreenivasan (1989) on the basis of a normalization requirement; they observed that without such a correction the singularity spectrum $f(\alpha)$ comes out wrong; they also pointed out that a similar correction has been proposed by van de Water & Schram (1988) in connection with the measurement of generalized Renyi dimensions.

Returning to the multifractal formalism of turbulence, beyond the random multiplicative model, we observe that the usual multifractal ansatz as made in Parisi & Frisch (1985) is only about the leading term of the probability, which is easily reinterpreted in geometrical language. Hence, it does not allow us to determine logarithmic corrections in structure functions. However, if we use Kolmogorov's four-fifths law, we have an additional piece of information which implies that the multifractal probability should have a subleading logarithmic correction with precisely the form it has in (4.1). This improved form then rules out subleading logarithmic corrections in any of the structure functions.

Finally, we should comment on those physical effects which we know for sure to be responsible for subleading corrections (log or not log) to isotropic scaling. This is an interesting question which we wish to briefly address. There is at least one known instance which has a genuine logarithm in its third-order structure function, namely the Burgers equation (in the limit of vanishing viscosity) with a Gaussian random force which is white in time and has a $1/k$ spatial spectrum, where k is the wavenumber. As shown by Chekhlov & Yakhot (1995) and Mitra *et al.* (2005), the Burgers equivalent of the four-fifths law implies the presence of a logarithmic correction. What was less obvious is that another frequently considered structure function, defined with the absolute value of the velocity increment, also has a logarithmic correction but accompanied by a subdominant term (proportional to the separation without a log factor) which conspires to make this structure function appear to have anomalous power-law scaling with a non-trivial exponent (Mitra *et al.* 2005). This is an artifact which would also be present in three-dimensional Navier–Stokes turbulence with $1/k$ forcing. A number of other artifacts which can hide the true scaling were reviewed at a recent workshop held in Beaulieu-sur-mer (see http://www.obs-nice.fr/etc7/anomalous).

Particularly noteworthy are the contaminations by subdominant terms stemming from anisotropy (see e.g. Biferale & Procaccia 2005).

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